Backward uniqueness for parabolic operators with non–Lipschitz coefficients

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Abstract

We investigate the relation between the backward uniqueness and the regularity of the coefficients for a parabolic operator. A necessary and sufficient condition for uniqueness is given in terms of the modulus of continuity of the coefficients.

Keywords: backward uniqueness, parabolic operators, modulus of continuity, Osgood condition

1 Introduction

We consider the following backward parabolic operator

$$L = \partial_t + \sum_{i,j=1}^{n} \partial_{x_j}(a_{jk}(t,x)\partial_{x_k}) + \sum_{j=1}^{n} b_j(t,x)\partial_{x_j} + c(t,x). \qquad (1.1)$$

All the coefficients are supposed to be defined in $[0,T]\times\mathbb{R}^n_x$, measurable and bounded; the coefficients b_j and c are complex valued; $(a_{jk}(t,x))_{jk}$ is a real symmetric matrix for all $(t,x)\in[0,T]\times\mathbb{R}^n_x$ and there exists $\lambda_0\in(0,1]$ such

that

$$\sum_{j,k=1}^{n} a_{jk}(t,x)\xi_j\xi_k \geq \lambda_0|\xi|^2$$

for all $(t,x) \in [0,T] \times \mathbb{R}^n_x$ and $\xi \in \mathbb{R}^n_\xi$.

Given a functional space $\mathcal{H}$ (in which it makes sense to look for the solutions of the equation $Lu = 0$) we say that the operator L has the $\mathcal{H}$–uniqueness property if, whenever $u \in \mathcal{H}$, $Lu = 0$ in $[0,T] \times \mathbb{R}^n_x$ and $u(0,x) = 0$ in $\mathbb{R}^n_x$, then $u = 0$ in $[0,T] \times \mathbb{R}^n_x$.

The problem we are interested in is the following: find the minimal regularity on the coefficients a_{jk} ensuring the $\mathcal{H}$–uniqueness property to L.

We remark that even in the simplest case (i. e. $(a_{jk})_{jk} = \mathrm{Id}$) the answer may depend on $\mathcal{H}$ and in particular on the rate of growth of u with respect to the x variables, as the classical example of Tychonoff [17] shows.

Considering $\mathcal{H}_1 = H^1([0,T], L^2(\mathbb{R}^n_x)) \cap L^2([0,T], H^2(\mathbb{R}^n_x))$, $\mathcal{H}_1$–uniqueness for L has been proved under the hypothesis of Lipschitz–continuity of the coefficients a_{jk} by Lions and Malgrange [12] (see for related or more general results [14], [1], [2], [11]). On the other hand the well known example of Miller [13] (where an operator having coefficients which are Hölder–continuous of order $1/6$ with respect to t and C^∞ with respect to x does not have the uniqueness property) shows that a certain amount of regularity on the a_{jk}'s is necessary for the $\mathcal{H}_1$–uniqueness.

The first part of the present work is devoted to prove the $\mathcal{H}_1$–uniqueness property for the operator (1.1) when the coefficients a_{jk} are C^2 in the x variables and non–Lipschitz–continuous in t. The regularity in t will be given in terms of a modulus of continuity μ satisfying the so called Osgood condition

$$\int_0^1 \frac{1}{\mu(s)}\, ds = +\infty.$$

This uniqueness result is a consequence of a Carleman estimate in which the weight function depends on the modulus of continuity; such kind of weight functions in Carleman estimates have been introduced by Tarama [16] in the case of second order elliptic operators. In obtaining our Carleman estimate the integrations by parts, which cannot be used since the coefficients are not Lipschitz–continuous, are replaced by a microlocal approximation procedure similar to the one exploited by Colombini and Lerner [8] to prove some energy estimates for hyperbolic operators with log–Lipschitz coefficients (see also [4] and [5]).

It is interesting to remark that the Osgood condition is also necessary for the $\mathcal{H}_1$–uniqueness property, at least when only the regularity in t of the coefficients a_{jk} is concerned. Precisely in the second part of this paper we prove that if a modulus of continuity does not satisfy the Osgood condition then it is possible to construct a backward parabolic operator of type (1.1) such that the coefficients a_{jk} depend only on t, the regularity of the a_{jk}'s is ruled by the modulus of continuity and the operator has not the $\mathcal{H}_1$–uniqueness property. The construction of this class of examples is modelled on a well known non–uniqueness result for elliptic operators due to Pliś [15].

The plan of the paper is the following: in Section 2 we give the precise statement of the uniqueness theorem and we present the non–uniqueness examples; a remark is devoted to compare these results with similar ones known for elliptic and hyperbolic operators. Section 3 contains the proof of the uniqueness results. In Section 4 we sketch the construction of the counter examples.

We denote by $\langle \cdot, \cdot \rangle_{L^2}$ the scalar product in $L^2(\mathbb{R}^n_x)$ and by $\| \cdot \|_{L^2}$ the corresponding norm. We denote by $\| \cdot \|_{\mathcal{B}}$ the norm of any other Banach space $\mathcal{B}$. Finally we denote by ∇ the gradient with respect to the x variables.

2 Results and remarks

Let μ be a modulus of continuity, i. e. let $\mu : [0,1] \to [0,1]$ be continuous, concave, strictly increasing, with $\mu(0) = 0$. Let $I \subseteq \mathbb{R}$ and let $\varphi : I \to \mathcal{B}$, where $\mathcal{B}$ is a Banach space. We say that $\varphi \in C^\mu(I, \mathcal{B})$ if $\varphi \in L^\infty(I, \mathcal{B})$ and

$$\sup_{\substack{0<|t-s|<1 \\ t,s\in I}} \frac{\|\varphi(t) - \varphi(s)\|_{\mathcal{B}}}{\mu(|t-s|)} < +\infty.$$

Remark 1. *The concavity of μ implies that $\mu(s) \geq s\mu(1)$ for all $s \in [0,1]$; the same reason makes the function $s \mapsto \frac{\mu(s)}{s}$ decreasing on $]0,1]$. Consequently there exists $\lim_{s\to 0^+} \frac{\mu(s)}{s}$. If $\sup_{s\in]0,1]} \frac{\mu(s)}{s} < +\infty$ then there exists $C > 0$ such that $\mu(s) \leq Cs$ for all $s \in [0,1]$ and hence $C^\mu = \mathrm{Lip}$. As a consequence, if $C^\mu \neq \mathrm{Lip}$, in particular if $\int_0^1 1/\mu(s)\, ds < +\infty$, then $\lim_{s\to 0^+} \frac{\mu(s)}{s} = +\infty$. Finally the function $\sigma \mapsto \mu(1/\sigma)/(1/\sigma)$ is increasing on $[1, +\infty[$; consequently the function $\sigma \mapsto \sigma^2\mu(1/\sigma)$ is increasing on $[1, +\infty[$ and the function $\sigma \mapsto 1/(\sigma^2\mu(1/\sigma))$ is decreasing on the same interval.*

We can now state our main uniqueness result.

Theorem 1. *Let μ be a modulus of continuity and suppose*

$$\int_0^1 \frac{1}{\mu(s)}\, ds = +\infty. \tag{2.1}$$

Suppose, for all $j, k = 1 \dots, n$, $a_{jk} \in C^\mu([0,T], C_b^2(\mathbb{R}_x^n))$ where $C_b^2(\mathbb{R}_x^n)$ is the space of twice differentiable functions which are bounded with bounded derivatives.

Then the operator L defined in (1.1) has the $\mathcal{H}_1$–uniqueness property.

Let us denote by $\mathcal{H}_2$ the space of functions w defined in $[0,T] \times \mathbb{R}_x^n$ such that w is continuous and differentiable with respect to t with continuous derivative and twice differentiable with respect to x with continuous derivatives and there exists $C > 0$ such that

$$|w(t,x)|,\ |\partial_t w(t,x)|,\ |\partial_{x_j} w(t,x)|,\ |\partial_{x_j}\partial_{x_k} w(t,x)| \le C e^{C|x|}$$

for all $j, k = 1 \dots, n$ and for all $(t,x) \in [0,T] \times \mathbb{R}_x^n$. The following result holds.

Theorem 2. *In the hypotheses of Theorem 1 the operator L has the $\mathcal{H}_2$–uniqueness property.*

The condition (2.1) on μ is known as "Osgood condition" (see e.g. [10, p. 160]. Our next result shows that this condition is necessary to have the uniqueness property.

Theorem 3. *Let μ be a modulus of continuity and suppose*

$$\int_0^1 \frac{1}{\mu(s)}\, ds < +\infty. \tag{2.2}$$

Then there exists $l \in C^\mu(\mathbb{R}_t)$ with $1/2 \le l(t) \le 3/2$ for all $t \in \mathbb{R}_t$ and there exists u, b_1, b_2, $c \in C_b^\infty(\mathbb{R}_t \times \mathbb{R}_x^2)$ with $\operatorname{supp} u = \{t \ge 0\}$ such that

$$\partial_t u + \partial_{x_1}^2 u + l\partial_{x_2}^2 u + b_1 \partial_{x_1} u + b_2 \partial_{x_2} u + cu = 0 \quad \text{in } \mathbb{R}_t \times \mathbb{R}_x^2. \tag{2.3}$$

Remark 2. *Considering a function $\theta \in C^\infty(\mathbb{R}_x^n)$ such that $\theta(x) = e^{-C|x|}$ for $|x| \ge 1$ and taking $v(t,x) = \theta(x)u(t,x)$ where $u(t,x)$ is the function constructed in Theorem 3, we immediately obtain a counter example to the $\mathcal{H}_1$–uniqueness result.*

Remark 3. *It may be interesting to compare the uniqueness and non–uniqueness results presented here with similar ones known for different classes of operators. The case of second order elliptic operators with real principal part has been considered by Tarama [16]. The uniqueness in the Cauchy problem is obtained for such kind of operators when the coefficients of the principal part are C^μ with respect to all the variables and μ satisfies the condition (2.1). A precise analysis of the non–uniqueness example of Pliś [15] shows that (2.1) is necessary (see [9]).*

An example of non–uniqueness for hyperbolic operators having the coefficients of the principal part in C^μ with μ satisfying the condition (2.2) is given in [6] (see also [7]). It is an open problem, whether (2.1) is sufficient to have the uniqueness in the Cauchy problem for second order hyperbolic operators.

3 Proofs of Theorems 1 and 2

In this paragraph we prove Theorem 1 and Theorem 2. Theorem 1 will follow in standard way from a Carleman estimate. In order to state the latter, we need first to introduce the weight function. We define

$$\phi(t) = \int_{\frac{1}{t}}^{1} \frac{1}{\mu(s)}\, ds.$$

The function ϕ is a strictly increasing C^1 function. From (2.1) we have $\phi([1,+\infty[) = [0,+\infty[$ and $\phi'(t) = 1/(t^2\mu(1/t)) > 0$ for all $t \in [1,+\infty[$. We set

$$\Phi(\tau) = \int_0^\tau \phi^{-1}(s)\, ds.$$

We obtain $\Phi'(\tau) = \phi^{-1}(\tau)$ and consequently $\lim_{\tau\to+\infty} \Phi'(\tau) = +\infty$. Moreover

$$\Phi''(\tau) = (\Phi'(\tau))^2\mu(\frac{1}{\Phi'(\tau)}) \tag{3.1}$$

for all $\tau \in [0,+\infty[$ and, as the function $\sigma \mapsto \sigma\mu(1/\sigma)$ is increasing on $[1,+\infty[$ (see Remark 1), we deduce that

$$\lim_{\tau\to+\infty} \Phi''(\tau) = \lim_{\tau\to+\infty} (\Phi'(\tau))^2\mu(\frac{1}{\Phi'(\tau)}) = +\infty. \tag{3.2}$$

Now we can state the Carleman estimate.

Proposition 1. *There exist* γ_0, $C > 0$ *such that*

$$\begin{aligned}\int_0^{\frac{T}{2}} e^{\frac{2}{\gamma}\Phi(\gamma(T-t))} \|\partial_t u + \sum_{jk} \partial_{x_j}(a_{jk}\partial_{x_k} u)\|_{L^2}^2 \, dt \\ \geq C\gamma^{\frac{1}{2}} \int_0^{\frac{T}{2}} e^{\frac{2}{\gamma}\Phi(\gamma(T-t))} (\|\nabla u\|_{L^2}^2 + \gamma^{\frac{1}{2}} \|u\|_{L^2}^2) \, dt\end{aligned} \tag{3.3}$$

for all $\gamma > \gamma_0$ *and for all* $u \in C_0^\infty(\mathbb{R}_t \times \mathbb{R}_x^n, \mathbb{C})$ *such that* $\operatorname{supp} u \subseteq [0, T/2] \times \mathbb{R}_x^n$.

The proof of the Proposition 1 is rather long and we divide it in several steps.

a) the Littlewood–Paley decomposition

We set $v(t,x) = e^{\frac{1}{\gamma}\Phi(\gamma(T-t))} u(t,x)$. The inequality (3.3) becomes

$$\begin{aligned}\int_0^{\frac{T}{2}} \|\partial_t v + \sum_{jk} \partial_{x_j}(a_{jk}\partial_{x_k} v) + \Phi'(\gamma(T-t))v\|_{L^2}^2 \, dt \\ \geq C\gamma^{\frac{1}{2}} \int_0^{\frac{T}{2}} (\|\nabla v\|_{L^2}^2 + \gamma^{\frac{1}{2}} \|v\|_{L^2}^2) \, dt.\end{aligned} \tag{3.4}$$

We use now the Littlewood–Paley decomposition technique. We recall some basic facts on it, referring to [3] and [8] for further details. Let $\varphi_0 \in C_0^\infty(\mathbb{R}_\xi^n)$, $0 \leq \varphi(\xi) \leq 1$ for all $\xi \in \mathbb{R}_\xi^n$, $\varphi_0(\xi) = 1$ for all ξ such that $|\xi| \leq 1$, $\varphi_0(\xi) = 0$ for all ξ such that $|\xi| \geq 2$ and φ_0 radially decreasing. For all $\nu \in \mathbb{N} \setminus \{0\}$ we define

$$\varphi_\nu(\xi) = \varphi_0(\frac{\xi}{2^\nu}) - \varphi_0(\frac{\xi}{2^{\nu-1}}).$$

For $u \in L^2(\mathbb{R}_x^n, \mathbb{C})$ we set

$$u_\nu(x) = \varphi_\nu(D)u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}_\xi^n} e^{\mathrm{i}x\xi} \varphi_\nu(\xi)\hat{u}(\xi) \, d\xi, \tag{3.5}$$

where $\hat{u}$ is the Fourier-Plancherel transform of u. We remark that (3.5) makes sense also for $u \in \mathcal{S}'(\mathbb{R}_x^n, \mathbb{C})$ if the last integral is interpreted as the inverse Fourier transform of $\varphi(\xi)\hat{u}(\xi)$. We have that there exists $K > 0$ such that

$$\frac{1}{K} \sum_\nu \|u_\nu\|_{L^2}^2 \leq \|u\|_{L^2}^2 \leq K \sum_\nu \|u_\nu\|_{L^2}^2 \tag{3.6}$$

for all $u \in L^2(\mathbb{R}^n_x, \mathbb{C})$. Consequently

$$
\begin{aligned}
&\int_0^{\frac{T}{2}} \|\partial_t v + \sum_{jk} \partial_{x_j}(a_{jk}\partial_{x_k} v) + \Phi'(\gamma(T-t))v\|^2_{L^2}\, dt \\
&\quad\geq \frac{1}{K}\int_0^{\frac{T}{2}} \sum_\nu \|\varphi_\nu(D)(\partial_t v + \sum_{jk} \partial_{x_j}(a_{jk}\partial_{x_k} v) + \Phi'(\gamma(T-t))v)\|^2_{L^2}\, dt \\
&\quad\geq \frac{1}{K}\int_0^{\frac{T}{2}} \sum_\nu \|\partial_t v_\nu + \sum_{jk} \partial_{x_j}(a_{jk}\partial_{x_k} v_\nu) + \Phi'(\gamma(T-t))v_\nu \\
&\qquad\qquad\qquad + \sum_{jk} \partial_{x_j}([\varphi_\nu,\ a_{jk}]\partial_{x_k} v)\|^2_{L^2}\, dt \\
&\quad\geq \frac{1}{K}\int_0^{\frac{T}{2}} \sum_\nu \|\partial_t v_\nu + \sum_{jk} \partial_{x_j}(a_{jk}\partial_{x_k} v_\nu) + \Phi'(\gamma(T-t))v_\nu\|^2_{L^2}\, dt \\
&\qquad\qquad - \frac{1}{K}\int_0^{\frac{T}{2}} \sum_\nu \|\sum_{jk} \partial_{x_j}([\varphi_\nu,\ a_{jk}]\partial_{x_k} v_\nu)\|^2_{L^2}\, dt
\end{aligned}
\tag{3.7}
$$

where $[\varphi_\nu,\ a_{jk}]w = \varphi_\nu(D)(a_{jk}w) - a_{jk}\varphi_\nu(D)w$.

b) the approximation procedure

We start to estimate

$$
\int_0^{\frac{T}{2}} \sum_\nu \|\partial_t v_\nu + \sum_{jk} \partial_{x_j}(a_{jk}\partial_{x_k} v_\nu) + \Phi'(\gamma(T-t))v_\nu\|^2_{L^2}\, dt.
$$

We obtain

$$
\begin{aligned}
&\int_0^{\frac{T}{2}} \sum_\nu \|\partial_t v_\nu + \sum_{jk} \partial_{x_j}(a_{jk}\partial_{x_k} v_\nu) + \Phi'(\gamma(T-t))v_\nu\|^2_{L^2}\, dt \\
&\quad = \int_0^{\frac{T}{2}} \sum_\nu (\|\partial_t v_\nu\|^2_{L^2} + \|\sum_{jk} \partial_{x_j}(a_{jk}\partial_{x_k} v_\nu) + \Phi'(\gamma(T-t))v_\nu\|^2_{L^2} \\
&\qquad + \gamma\Phi''(\gamma(T-t))\|v_\nu\|^2_{L^2} + 2\,\mathrm{Re}\,\langle \partial_t v_\nu,\ \sum_{jk} \partial_{x_j}(a_{jk}\partial_{x_k} v_\nu)\rangle_{L^2})\, dt.
\end{aligned}
\tag{3.8}
$$

We remark that if a_{jk} would be Lipschitz-continuous the last term in (3.8) would be easily computed by integration by parts. On the contrary here we approximate it using a technique similar to the one of [4] (see also [8] and

[5]). Let $\rho \in C_0^\infty(\mathbb{R})$ with $\operatorname{supp}\rho \subseteq [-1/2,\ 1/2]$, $\int_{\mathbb{R}} \rho(s)\ ds = 1$ and $\rho(s) \geq 0$ for all $s \in \mathbb{R}$; we set

$$a_{jk,\,\varepsilon}(t,x) = \int_{\mathbb{R}} a_{jk}(s,x)\frac{1}{\varepsilon}\rho(\frac{t-s}{s})\ ds$$

for $\varepsilon \in\,]0,1/2]$. We obtain that there exist C, $\tilde{C} > 0$ such that

$$|a_{jk,\,\varepsilon}(t,x) - a_{jk}(t,x)| \leq C\mu(\varepsilon) \tag{3.9}$$

and

$$|\partial_t a_{jk,\,\varepsilon}(t,x)| \leq \tilde{C}\,\frac{\mu(\varepsilon)}{\varepsilon} \tag{3.10}$$

for all $j,k = 1\ldots,n$ and for all $(t,x) \in [0,T] \times \mathbb{R}^n_x$. We have

$$\begin{aligned}
&\int_0^{\frac{T}{2}} 2\operatorname{Re}\langle \partial_t v_\nu,\ \sum_{jk} \partial_{x_j}(a_{jk}\partial_{x_k} v_\nu)\rangle_{L^2}\ dt \\
&\quad = -2\operatorname{Re}\int_0^{\frac{T}{2}} \sum_{jk} \langle \partial_{x_j}\partial_t v_\nu,\ a_{jk}\partial_{x_k} v_\nu\rangle_{L^2}\ dt \\
&\quad = -2\operatorname{Re}\int_0^{\frac{T}{2}} \sum_{jk} \langle \partial_{x_j}\partial_t v_\nu,\ (a_{jk} - a_{jk,\,\varepsilon})\partial_{x_k} v_\nu\rangle_{L^2}\ dt \\
&\qquad\qquad -2\operatorname{Re}\int_0^{\frac{T}{2}} \sum_{jk} \langle \partial_{x_j}\partial_t v_\nu,\ a_{jk,\,\varepsilon}\partial_{x_k} v_\nu\rangle_{L^2}\ dt.
\end{aligned}$$

We remark that $\|\partial_{x_j} v_\nu\|_{L^2} \leq 2^{\nu+1}\|v_\nu\|_{L^2}$ and $\|\partial_{x_j}\partial_t v_\nu\|_{L^2} \leq 2^{\nu+1}\|\partial_t v_\nu\|_{L^2}$ for all $\nu \in \mathbb{N}$ so that from (3.9) we get

$$\begin{aligned}
&|2\operatorname{Re}\int_0^{\frac{T}{2}} \sum_{jk} \langle \partial_{x_j}\partial_t v_\nu,\ (a_{jk} - a_{jk,\,\varepsilon})\partial_{x_k} v_\nu\rangle_{L^2}\ dt| \\
&\quad \leq C\mu(\varepsilon)\int_0^{\frac{T}{2}} \sum_{jk} \|\partial_{x_j}\partial_t v_\nu\|_{L^2}\ \|\partial_{x_k} v_\nu\|_{L^2}\ dt \\
&\qquad \leq \frac{n^2 C}{N}\int_0^{\frac{T}{2}} \|\partial_t v_\nu\|^2_{L^2}\ dt + n^2 CN\, 2^{4(\nu+1)}\mu(\varepsilon)\int_0^{\frac{T}{2}} \|v_\nu\|^2_{L^2}\ dt
\end{aligned}$$

for all $N > 0$, and similarly from (3.10) we deduce

$$|2\,\mathrm{Re}\int_0^{\frac{T}{2}}\sum_{jk}\langle\partial_{x_j}\partial_t v_\nu,\ a_{jk,\varepsilon}\partial_{x_k}v_\nu\rangle_{L^2}\ dt| = |\int_0^{\frac{T}{2}}\sum_{jk}\langle\partial_{x_j}v_\nu,\ \partial_t a_{jk,\varepsilon}\partial_{x_k}v_\nu\rangle_{L^2}\ dt|$$
$$\le n\tilde{C}\,\frac{\mu(\varepsilon)}{\varepsilon}\int_0^{\frac{T}{2}}\|\nabla v_\nu\|_{L^2}^2\ dt \le n^2\tilde{C}\,2^{2(\nu+1)}\,\frac{\mu(\varepsilon)}{\varepsilon}\int_0^{\frac{T}{2}}\|v_\nu\|_{L^2}^2\ dt.$$

Let $N = n^2C$. We deduce that, for all $\nu \in \mathbb{N}$,

$$\begin{aligned}\int_0^{\frac{T}{2}}&\|\partial_t v_\nu + \sum_{jk}\partial_{x_j}(a_{jk}\partial_{x_k}v_\nu) + \Phi'(\gamma(T-t))v_\nu\|_{L^2}^2\ dt\\ &\ge \int_0^{\frac{T}{2}}(\|\sum_{jk}\partial_{x_j}(a_{jk}\partial_{x_k}v_\nu) + \Phi'(\gamma(T-t))v_\nu\|_{L^2}^2 + \gamma\Phi''(\gamma(T-t))\|v_\nu\|_{L^2}^2\\ &\qquad -(n^4C^2\,2^{4(\nu+1)}\,\mu(\varepsilon) + n^2\tilde{C}\,2^{2(\nu+1)}\,\frac{\mu(\varepsilon)}{\varepsilon})\|v_\nu\|_{L^2}^2)\ dt.\end{aligned} \tag{3.11}$$

Let $\nu = 0$. From (3.2) we can choose $\gamma_0 > 0$ such that $\Phi''(\gamma(T-t)) \ge 1$ for all $\gamma > \gamma_0$ and for all $t \in [0,\ T/2]$. Taking now $\varepsilon = 1/2$ we obtain from (3.11) that

$$\begin{aligned}\int_0^{\frac{T}{2}}&\|\partial_t v_0 + \sum_{jk}\partial_{x_j}(a_{jk}\partial_{x_k}v_0) + \Phi'(\gamma(T-t))v_0\|_{L^2}^2\ dt\\ &\ge \int_0^{\frac{T}{2}}(\gamma - 8n^2\mu(\frac{1}{2})(2n^2C^2 + \tilde{C}))\|v_0\|_{L^2}^2\ dt\end{aligned}$$

for all $\gamma > \gamma_0$. Possibly choosing a larger γ_0 we have, again for all $\gamma > \gamma_0$,

$$\begin{aligned}\int_0^{\frac{T}{2}}&\|\partial_t v_0 + \sum_{jk}\partial_{x_j}(a_{jk}\partial_{x_k}v_0) + \Phi'(\gamma(T-t))v_0\|_{L^2}^2\ dt\\ &\ge \frac{\gamma}{2}\int_0^{\frac{T}{2}}\|v_0\|_{L^2}^2\ dt.\end{aligned} \tag{3.12}$$

Let now $\nu \ge 1$. We recall that in this case $\|\nabla v_\nu\| \ge 2^{\nu-1}\|v_\nu\|$. We take

$\varepsilon = 2^{-2\nu}$. We obtain from (3.11) that

$$
\begin{aligned}
&\int_0^{\frac{T}{2}} \|\partial_t v_\nu + \sum_{jk} \partial_{x_j}(a_{jk}\partial_{x_k} v_\nu) + \Phi'(\gamma(T-t))v_\nu\|_{L^2}^2 \, dt \\
&\qquad \geq \int_0^{\frac{T}{2}} (\| \sum_{jk} \partial_{x_j}(a_{jk}\partial_{x_k} v_\nu) + \Phi'(\gamma(T-t))v_\nu\|_{L^2}^2 \\
&\qquad\qquad\qquad + \gamma \Phi''(\gamma(T-t))\|v_\nu\|_{L^2}^2 - K\, 2^{4\nu}\, \mu(2^{-2\nu})\|v_\nu\|_{L^2}^2)\, dt \\
&\qquad \geq \int_0^{\frac{T}{2}} ((\| \sum_{jk} \partial_{x_j}(a_{jk}\partial_{x_k} v_\nu)\|_{L^2} - \Phi'(\gamma(T-t))\|v_\nu\|_{L^2})^2 \\
&\qquad\qquad\qquad + \gamma \Phi''(\gamma(T-t))\|v_\nu\|_{L^2}^2 - K\, 2^{4\nu}\, \mu(2^{-2\nu})\|v_\nu\|_{L^2}^2)\, dt
\end{aligned}
\tag{3.13}
$$

where $K = 16n^4C^2 + 4n^2\tilde{C}$. On the other hand we have

$$
\begin{aligned}
&\| \sum_{jk} \partial_{x_j}(a_{jk}\partial_{x_k} v_\nu)\|_{L^2} \, \|v_\nu\|_{L^2} \geq |\langle \sum_{jk} \partial_{x_j}(a_{jk}\partial_{x_k} v_\nu),\ v_\nu\rangle_{L^2}| \\
&\qquad\qquad \geq |\sum_{jk} \langle a_{jk}\partial_{x_k} v_\nu,\ \partial_{x_j} v_\nu\rangle_{L^2}| \geq \lambda_0 \|\nabla v_\nu\|_{L^2}^2 \geq \frac{\lambda_0}{4}\, 2^{2\nu}\, \|v_\nu\|_{L^2}^2
\end{aligned}
$$

and consequently

$$
\| \sum_{jk} \partial_{x_j}(a_{jk}\partial_{x_k} v_\nu)\|_{L^2} \geq \frac{\lambda_0}{4}\, 2^{2\nu}\, \|_{L^2} v_\nu\|. \tag{3.14}
$$

Suppose first that $\Phi'(\gamma(T-t)) \leq \frac{\lambda}{8}\, 2^{2\nu}$. Then from (3.14) we deduce that

$$
\| \sum_{jk} \partial_{x_j}(a_{jk}\partial_{x_k} v_\nu)\|_{L^2} - \Phi'(\gamma(T-t))\|v_\nu\|_{L^2} \geq \frac{\lambda}{8}\, 2^{2\nu} \|v_\nu\|_{L^2}
$$

and then, using also the fact that $\Phi''(\gamma(T-t)) \geq 1$, we obtain that there

exist γ_0 and $c>0$ such that

$$\begin{aligned}
&\int_0^{\frac{T}{2}} ((\| \sum_{jk} \partial_{x_j}(a_{jk}\partial_{x_k} v_\nu)\|_{L^2} - \Phi'(\gamma(T-t))\|v_\nu\|_{L^2})^2 \\
&\qquad\qquad + \gamma\Phi''(\gamma(T-t))\|v_\nu\|^2_{L^2} - K\, 2^{4\nu}\,\mu(2^{-2\nu})\|v_\nu\|^2_{L^2}\,)dt \\
&\quad \geq \int_0^{\frac{T}{2}} ((\frac{\lambda}{8}\,2^{2\nu})^2 + \gamma - K\,2^{4\nu}\,\mu(2^{-2\nu})\|v_\nu\|^2_{L^2})\, dt \\
&\qquad \geq \int_0^{\frac{T}{2}} ((\frac{\lambda}{16})^2\, 2^{4\nu} + \frac{2}{3}\gamma)\|v_\nu\|^2_{L^2}\, dt \geq \int_0^{\frac{T}{2}} (\frac{\gamma}{2} + c\gamma^{\frac{1}{2}}\, 2^{2\nu})\|v_\nu\|^2_{L^2}\, dt
\end{aligned} \tag{3.15}$$

for all $\gamma \geq \gamma_0$. If on the contrary $\Phi'(\gamma(T-t)) \geq \frac{\lambda}{8}\,2^{2\nu}$ then, using (3.1), the fact that $\lambda_0 \leq 1$ and the properties of μ (see Remark 1),

$$\begin{aligned}
\Phi''(\gamma(T-t)) &= (\Phi'(\gamma(T-t))^2 \mu(\frac{1}{\Phi'(\gamma(T-t))}) \\
&\geq (\frac{\lambda_0}{8})^2\, 2^{4\nu}\, \mu(\frac{8}{\lambda_0}\, 2^{-2\nu}) \geq (\frac{\lambda_0}{8})^2\, 2^{4\nu}\, \mu(2^{-2\nu}).
\end{aligned}$$

Hence also in this case there exist γ_0 and $c>0$ such that

$$\begin{aligned}
&\int_0^{\frac{T}{2}} ((\| \sum_{jk} \partial_{x_j}(a_{jk}\partial_{x_k} v_\nu)\|_{L^2} - \Phi'(\gamma(T-t))\|v_\nu\|_{L^2})^2 \\
&\qquad\qquad + \gamma\Phi''(\gamma(T-t))\|v_\nu\|^2_{L^2} - K\, 2^{4\nu}\,\mu(2^{-2\nu})\|v_\nu\|^2_{L^2})\, dt \\
&\quad \geq \int_0^{\frac{T}{2}} (\frac{\gamma}{2} + (\frac{\gamma}{2}(\frac{\lambda}{8})^2 - K)\, 2^{4\nu}\mu(2^{-2\nu}))\|v_\nu\|^2_{L^2}\, dt \\
&\qquad \geq \int_0^{\frac{T}{2}} (\frac{\gamma}{2} + c\gamma\, 2^{2\nu})\|v_\nu\|^2_{L^2}\, dt
\end{aligned} \tag{3.16}$$

for all $\gamma \geq \gamma_0$. Putting together (3.15) and (3.16) we have that there exist γ_0 and $c>0$ such that

$$\begin{aligned}
&\int_0^{\frac{T}{2}} \|\partial_t v_\nu + \sum_{jk} \partial_{x_j}(a_{jk}\partial_{x_k} v_\nu) + \Phi'(\gamma(T-t))v_\nu\|^2_{L^2}\, dt \\
&\qquad \geq \int_0^{\frac{T}{2}} (\frac{\gamma}{2} + c\gamma^{\frac{1}{2}}\, 2^{2\nu})\|v_\nu\|^2_{L^2}\, dt
\end{aligned} \tag{3.17}$$

for all $\nu \geq 1$ and for all $\gamma \geq \gamma_0$.

Form (3.12) and (3.17) we get that there exist γ_0 and $\tilde{c} > 0$ such that

$$\begin{aligned}\int_0^{\frac{T}{2}} \sum_\nu \|\partial_t v_\nu + \sum_{jk} \partial_{x_j}(a_{jk}\partial_{x_k} v_\nu) + \Phi'(\gamma(T-t))v_\nu\|_{L^2}^2 \, dt \\ \geq \tilde{c}\gamma^{\frac{1}{2}} \int_0^{\frac{T}{2}} \sum_\nu (\gamma^{\frac{1}{2}}\|v_\nu\|_{L^2}^2 + \|\nabla v_\nu\|_{L^2}^2) \, dt\end{aligned} \tag{3.18}$$

for all $\gamma \geq \gamma_0$.

c) the estimate for the commutator

For $\psi \in C_0^\infty(\mathbb{R}^n_\xi)$, we define $\check{\psi}(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n_\xi} e^{\mathrm{i}x\xi}\psi(\xi)\, d\xi$. Notice that $(\nabla\psi)\check{}(x) = i\check{\psi}(x)x$ and $(\psi_1\psi_2)\check{} = \check{\psi}_1 * \check{\psi}_2$. For $w \in L^2(\mathbb{R}^n_x, \mathbb{C})$ we have

$$w_\nu(x) = \int_{\mathbb{R}^n_y} \check{\varphi}_\nu(x-y)w(y)\, dy.$$

Moreover

$$[\varphi_\nu,\ a_{jk}]w(x) = \int_{\mathbb{R}^n_y} h^\nu_{jk}(x,y)w(y)\, dy,$$

where

$$h^\nu_{jk}(x,y) = \check{\varphi}_\nu(x-y)(a_{jk}(y) - a_{jk}(x))$$

(to avoid cumbersome notations here and throughout this point we drop t in writing the variables of the coefficients a_{jk}). One can rewrite h^ν_{jk} as $h^\nu_{jk}(x,y) = h^{\nu,1}_{jk}(x,y) + h^{\nu,2}_{jk}(x,y)$, where

$$\begin{aligned}h^{\nu,1}_{jk}(x,y) &= \check{\varphi}_\nu(x-y)\int_0^1 (\nabla a_{jk}(x+\theta(y-x)) - \nabla a_{jk}(x))\cdot(y-x)\, d\theta \\ h^{\nu,2}_{jk}(x,y) &= \check{\varphi}_\nu(x-y)\nabla a_{jk}(x)\cdot(y-x).\end{aligned}$$

We remark that

$$\int_{\mathbb{R}^n_y} h^{\nu,2}_{jk}(x,y)w(y)\, dy = \sum_{\mu=0}^{+\infty}\int_{\mathbb{R}^n_y} h^{\nu,2}_{jk}(x,y)w_\mu(y)\, dy,$$

where $w_\mu(x) = \varphi_\mu(D)w(x)$. We have then

$$\begin{aligned}
&\int_{\mathbb{R}^n_y} h^{\nu,2}_{jk}(x,y)w_\mu(y)\,dy \\
&\qquad = \int_{\mathbb{R}^n_y} \check{\varphi}_\nu(x-y)\nabla a_{jk}(x)\cdot(y-x)\Big(\int_{\mathbb{R}^n_z} \check{\varphi}_\mu(y-z)w(z)\,dz\Big)\,dy \\
&\qquad = \int_{\mathbb{R}^n_z} \nabla a_{jk}(x)\cdot\Big(\int_{\mathbb{R}^n_y} \check{\varphi}_\mu(y-z)\check{\varphi}_\nu(x-y)\,dy\Big)w(z)\,dz \\
&\qquad = \int_{\mathbb{R}^n_z} \nabla a_{jk}(x)\cdot\Big(\int_{\mathbb{R}^n_y} \mathrm{i}\check{\varphi}_\mu(y-z)(\nabla\varphi_\nu)\check{}\,(x-y)\,dy\Big)w(z)\,dz \\
&\qquad = \int_{\mathbb{R}^n_z} \nabla a_{jk}(x)\cdot\Big(\int_{\mathbb{R}^n_y} \mathrm{i}\check{\varphi}_\mu(y)(\nabla\varphi_\nu)\check{}\,((x-z)-y)\,dy\Big)w(z)\,dz.
\end{aligned}$$

Recalling that if $\mu < \nu - 1$ or $\mu > \nu + 1$ then

$$\int_{\mathbb{R}^n_y} \check{\varphi}_\mu(y)(\nabla\varphi_\nu)\check{}\,((x-z)-y)\,dy = (\varphi_\mu\nabla\varphi_\nu)\check{}\,(x-z) = 0,$$

we finally obtain

$$\int_{\mathbb{R}^n_y} h^{\nu,2}_{jk}(x,y)w(y)\,dy = \int_{\mathbb{R}^n_y} h^{\nu,2}_{jk}(x,y)(w_{\nu-1}(y)+w_\nu(y)+w_{\nu+1}(y))\,dy,$$

where we have set $w_{-1} = 0$ identically. We deduce

$$\begin{aligned}
&\partial_{x_l}[\varphi_\nu\; a_{jk}]w(x) \\
&\qquad = \int_{\mathbb{R}^n_y} \partial_{x_l}h^{\nu,1}_{jk}(x,y)w(y)\,dy \\
&\qquad\quad + \int_{\mathbb{R}^n_y} \partial_{x_l}h^{\nu,2}_{jk}(x,y)(w_{\nu-1}(y)+w_\nu(y)+w_{\nu+1}(y))\,dy.
\end{aligned} \tag{3.19}$$

Using the explicit expression of $h^{\nu,1}_{jk}$ we get

$$\begin{aligned}
&\partial_{x_l}h^{\nu,1}_{jk}(x,y) \\
&= \partial_{x_l}\check{\varphi}_\nu(x-y)\int_0^1 (\nabla a_{jk}(x+\theta(y-x)) - \nabla a_{jk}(x))\cdot(y-x)\,d\theta \\
&\quad + \check{\varphi}_\nu(x-y)\int_0^1 ((1-\theta)\nabla(\partial_{x_l}a_{jk})(x+\theta(y-x)) - \nabla(\partial_{x_l}a_{jk})(x))\cdot(y-x)\,d\theta \\
&\quad\quad - \check{\varphi}_\nu(x-y)\int_0^1 (\partial_{x_l}a_{jk}(x+\theta(y-x)) - \partial_{x_l}a_{jk}(x))\,d\theta.
\end{aligned}$$

Using the mean value theorem we deduce that

$$|\partial_{x_l} h^{\nu,1}_{jk}(x,y)| \le (|\partial_{x_l}\check{\varphi}_\nu(x-y)||x-y|^2 + 3|\check{\varphi}_\nu(x-y)||x-y|)\|D^2 a_{jk}\|_{L^\infty}.$$

Hence both $\int_{\mathbb{R}^n_x} |\partial_{x_l} h^{\nu,1}_{jk}(x,y)|\,dx$ and $\int_{\mathbb{R}^n_y} |\partial_{x_l} h^{\nu,1}_{jk}(x,y)|\,dy$ are dominated by the quantity

$$\|D^2 a_{jk}\|_{L^\infty} \int_{\mathbb{R}^n} (|\partial_{x_l}\check{\varphi}_\nu(z)||z|^2 + 3|\check{\varphi}_\nu(z)||z|)\,dz. \tag{3.20}$$

Now we observe that, for $\nu \ge 1$,

$$\check{\varphi}_\nu(z) = \check{\varphi}(2^\nu z)\,2^{n\nu} \quad \text{and} \quad \partial_{x_l}\check{\varphi}_\nu(z) = \partial_{x_l}\check{\varphi}(2^\nu z)\,2^{(n+1)\nu}, \tag{3.21}$$

where $\varphi(\xi) = \varphi_0(\xi) - \varphi_0(2\xi)$. Setting $2^\nu z = \zeta$, the quantity in (3.20) becomes

$$2^{-\nu}\|D^2 a_{jk}\|_{L^\infty} \int_{\mathbb{R}^n_\zeta} (|\partial_{x_l}\check{\varphi}(\zeta)||\zeta|^2 + 3|\check{\varphi}(\zeta)||\zeta|)\,d\zeta.$$

Consequently there exists $K > 0$ such that, for all $\nu \ge 0$,

$$\| \int_{\mathbb{R}^n_y} \partial_{x_l} h^{\nu,1}_{jk}(\cdot,y) w(y)\,dy\|_{L^2} \le K 2^{-\nu}\|w\|_{L^2}. \tag{3.22}$$

Next we consider

$$\begin{aligned}\partial_{x_l} h^{\nu,2}_{jk}(x,y) = \partial_{x_l}\check{\varphi}_\nu(x-y)\nabla a_{jk}(x)\cdot(y-x)&\\ +\check{\varphi}_\nu(x-y)\nabla(\partial_{x_l} a_{jk})(x)\cdot(y-x)&\\ +\check{\varphi}_\nu(x-y)\partial_{x_l} a_{jk}(x).&\end{aligned}$$

Again both $\int_{\mathbb{R}^n_x} |\partial_{x_l} h^{\nu,2}_{jk}(x,y)|\,dx$ and $\int_{\mathbb{R}^n_y} |\partial_{x_l} h^{\nu,2}_{jk}(x,y)|\,dy$ are dominated by

$$\begin{aligned}&\|\nabla a_{jk}\|_{L^\infty} \int_{\mathbb{R}^n_z} |\partial_{x_l}\check{\varphi}_\nu(z)||z|\,dz + \|D^2 a_{jk}\|_{L^\infty}\int_{\mathbb{R}^n_z} |\check{\varphi}_\nu(z)||z|\,dz\\ &\quad +\|\nabla a_{jk}\|_{L^\infty}\int_{\mathbb{R}^n_z} |\check{\varphi}_\nu(z)||z|\,dz.\end{aligned} \tag{3.23}$$

As before setting $2^\nu z = \zeta$ and recalling (3.21) we have that there exists $K > 0$ such that, for all $\nu \ge 0$,

$$\| \int_{\mathbb{R}^n_y} \partial_{x_l} h^{\nu,2}_{jk}(\cdot,y) w(y)\,dy\|_{L^2} \le K\|w\|_{L^2}. \tag{3.24}$$

It follows from (3.19), (3.22) and (3.24) that

$$\|\partial_{x_l}[\varphi_\nu,\, a_{jk}]w\|_{L^2} \le K(2^{-\nu}\,\|w\|_{L^2} + \|w_{\nu-1}\|_{L^2} + \|w_\nu\|_{L^2} + \|w_{\nu+1}\|_{L^2})$$

for all $\nu \ge 0$. Hence, possibly choosing a larger $K > 0$,

$$\begin{aligned}\|\partial_{x_j}[\varphi_\nu,\, a_{jk}]\partial_{x_k}v\|_{L^2} \le K(2^{-\nu}\|\partial_{x_k}v\|_{L^2} &+ \|(\partial_{x_k}v)_{\nu-1}\|_{L^2} \\ &+\|(\partial_{x_k}v)_\nu\|_{L^2} + \|(\partial_{x_k}v)_{\nu+1}\|_{L^2})\end{aligned}$$

for all $j,\ k = 1,\ldots,n,\ \nu \ge 0$ and $v \in C_0^\infty(\mathbb{R}^n,\mathbb{C})$. Finally from (3.6) we obtain that there exists a $\tilde{K}$ such that

$$\sum_\nu \|\sum_{jk}\partial_{x_j}[\varphi_\nu,\, a_{jk}]\partial_{x_k}v\|_{L^2}^2 \le \tilde{K}\|\nabla v\|_{L^2}^2. \tag{3.25}$$

d) end of the proof of Proposition 1

From (3.7), (3.18) and (3.25) we obtain that there exist γ_0, $\tilde{c}$, K and $\tilde{K}$ positive constants such that

$$\begin{aligned}&\int_0^{\frac{T}{2}} \|\partial_t v + \sum_{jk}\partial_{x_j}(a_{jk}\partial_{x_k}v) + \Phi'(\gamma(T-t))v\|_{L^2}^2\, dt \\ &\qquad \ge \frac{\tilde{c}}{K}\gamma^{\frac{1}{2}}\int_0^{\frac{T}{2}}\sum_\nu(\|\nabla v_\nu\|_{L^2}^2 + \gamma^{\frac{1}{2}}\|v_\nu\|_{L^2}^2)\, dt - \frac{\tilde{K}}{K}\int_0^{\frac{T}{2}}\|\nabla v\|_{L^2}^2\, dt\end{aligned}$$

for all $v \in C_0^\infty(\mathbb{R}_t \times \mathbb{R}_x^n,\mathbb{C})$ with support in $[0,T/2]\times\mathbb{R}_x^n$ and for all $\gamma \ge \gamma_0$. Using (3.6) we immediately obtain (3.4) and the proof of the Proposition 1 is complete.

Let us come finally to the proof of Theorem 1. First of all we remark that a density argument ensures that the inequality (3.3) holds for all $\gamma \ge \gamma_0$ and for all $u \in \mathcal{H}_1$ such that $\operatorname{supp} u \subseteq [0,T/2]\times\mathbb{R}_x^n$. Suppose now that $u \in \mathcal{H}_1$, $u(0,x) = 0$ in $\mathbb{R}_x^n$ and

$$\|\partial_t u + \sum_{jk}\partial_{x_j}(a_{jk}\partial_{x_k}u)\|_{L^2}^2 \le \tilde{C}(\|\nabla u\|_{L^2}^2 + \|u\|_{L^2}^2) \tag{3.26}$$

for a.e. $t \in [0,T]$. We consider $\omega \in C^\infty(\mathbb{R}_t)$ such that $\omega(t) = 0$ for all $t \ge T/2$ and $\omega(t) = 1$ for all $t \le T/3$. We apply (3.3) to the function $\omega(t)u(t,x)$ and

we obtain

$$\int_0^{\frac{T}{2}} e^{\frac{2}{\gamma}\Phi(\gamma(T-t))} \|\partial_t(\omega u) + \sum_{jk} \partial_{x_j}(a_{jk}\partial_{x_k}(\omega u))\|_{L^2}^2 \, dt$$
$$\geq C\gamma^{\frac{1}{2}} \int_0^{\frac{T}{2}} e^{\frac{2}{\gamma}\Phi(\gamma(T-t))} (\|\nabla(\omega u)\|_{L^2}^2 + \gamma^{\frac{1}{2}} \|\omega u\|_{L^2}^2) \, dt$$

and consequently

$$\int_0^{\frac{T}{3}} e^{\frac{2}{\gamma}\Phi(\gamma(T-t))} \|\partial_t u + \sum_{jk} \partial_{x_j}(a_{jk}\partial_{x_k} u)\|_{L^2}^2 \, dt$$
$$+ \int_{\frac{T}{3}}^{\frac{T}{2}} e^{\frac{2}{\gamma}\Phi(\gamma(T-t))} \|\partial_t(\omega u) + \sum_{jk} \partial_{x_j}(a_{jk}\partial_{x_k}(\omega u))\|_{L^2}^2 \, dt$$
$$\geq C\gamma^{\frac{1}{2}} \int_0^{\frac{T}{3}} e^{\frac{2}{\gamma}\Phi(\gamma(T-t))} (\|\nabla u\|_{L^2}^2 + \gamma^{\frac{1}{2}} \|u\|_{L^2}^2) \, dt.$$

By (3.26) we get

$$\int_{\frac{T}{3}}^{\frac{T}{2}} e^{\frac{2}{\gamma}\Phi(\gamma(T-t))} \|\partial_t(\omega u) + \sum_{jk} \partial_{x_j}(a_{jk}\partial_{x_k}(\omega u))\|_{L^2}^2 \, dt$$
$$\geq \int_0^{\frac{T}{3}} e^{\frac{2}{\gamma}\Phi(\gamma(T-t))} ((C\gamma^{\frac{1}{2}} - \tilde{C})\|\nabla u\|_{L^2}^2 + (C\gamma - \tilde{C})\|u\|_{L^2}^2) \, dt,$$

so that, since Φ is increasing,

$$e^{\frac{2}{\gamma}\Phi(\frac{2}{3}\gamma T)} \int_{\frac{T}{3}}^{\frac{T}{2}} \|\partial_t(\omega u) + \sum_{jk} \partial_{x_j}(a_{jk}\partial_{x_k}(\omega u))\|_{L^2}^2 \, dt$$
$$\geq e^{\frac{2}{\gamma}\Phi(\frac{3}{4}\gamma T)} \int_0^{\frac{T}{4}} ((C\gamma^{\frac{1}{2}} - \tilde{C})\|\nabla u\|_{L^2}^2 + (C\gamma - \tilde{C})\|u\|_{L^2}^2) \, dt.$$

Choosing γ_0 sufficiently large we deduce that for all $\gamma \geq \gamma_0$,

$$\int_{\frac{T}{3}}^{\frac{T}{2}} \|\partial_t(\omega u) + \sum_{jk} \partial_{x_j}(a_{jk}\partial_{x_k}(\omega u))\|_{L^2}^2 \, dt$$
$$\geq \frac{C}{2}\gamma e^{\frac{2}{\gamma}(\Phi(\frac{3}{4}\gamma T) - \Phi(\frac{2}{3}\gamma T))} \int_0^{\frac{T}{4}} \|u\|_{L^2}^2 \, dt.$$

Remarking now that

$$\lim_{\gamma\to+\infty}\frac{2}{\gamma}(\Phi(\frac{3}{4}\gamma T)-\Phi(\frac{2}{3}\gamma T))=\lim_{\gamma\to+\infty}\frac{2}{\gamma}\int_{\frac{2}{3}\gamma T}^{\frac{3}{4}\gamma T}\phi^{-1}(\tau)\,d\tau=+\infty,$$

we let γ go to $+\infty$ and we deduce that $u(t,x)=0$ in $[0,T/4]\times\mathbb{R}^n_x$. The conclusion of the proof of the Theorem 1 easily follows.

To prove Theorem 2 it will be sufficient to multiply u by a function $\theta\in C^\infty(\mathbb{R}^n_x)$ such that $\theta>0$ and $\theta(x)=e^{-2C|x|}$ for all $x\in\mathbb{R}^n_x$ with $|x|\geq 1$. A direct computation shows that $\theta u\in\mathcal{H}_1$ and satisfies (3.26). Consequently $\theta u=0$ in $[0,T]\times\mathbb{R}^n_x$ and the same will be for u.

4 Sketch of the proof of Theorem 3

In the proof of Theorem 3 we will follow closely the construction of the example in [15]. Let A, B, C, J be four C^∞ functions defined in $\mathbb{R}$ with $0\leq A(s),\ B(s),\ C(s)\leq 1$, $-2\leq J(s)\leq 2$ for all $s\in\mathbb{R}$ and

$$\begin{array}{ll}
A(s)=1\quad\text{for } s\leq\dfrac{1}{5}, & A(s)=0\quad\text{for } s\geq\dfrac{1}{4},\\
B(s)=0\quad\text{for } s\leq 0 \text{ or } s\geq 1, & B(s)=1\quad\text{for } \dfrac{1}{6}\leq s\leq\dfrac{1}{2},\\
C(s)=0\quad\text{for } s\leq\dfrac{1}{4}, & C(s)=1\quad\text{for } s\geq\dfrac{1}{3},\\
J(s)=-2\quad\text{for } s\leq\dfrac{1}{6} \text{ or } s\geq\dfrac{1}{2}, & J(s)=2\quad\text{for } \dfrac{1}{5}\leq s\leq\dfrac{1}{3}.
\end{array}$$

Let $(a_n)_n$, $(z_n)_n$ be two real sequences such that

$$-1<a_n<a_{n+1}\quad\text{for all } n\geq 1,\qquad \lim_n a_n=0, \tag{4.1}$$

$$1<z_n<z_{n+1}\quad\text{for all } n\geq 1,\qquad \lim_n z_n=+\infty; \tag{4.2}$$

and let us define $r_n=a_{n+1}-a_n$, $q_1=0$, $q_n=\sum_{k=2}^n z_k r_{k-1}$ for all $n\geq 2$, and $p_n=(z_{n+1}-z_n)r_n$. We suppose moreover that

$$p_n>1\quad\text{for all } n\geq 1. \tag{4.3}$$

We set $A_n(t) = A(\frac{t-a_n}{r_n})$, $B_n(t) = B(\frac{t-a_n}{r_n})$, $C_n(t) = C(\frac{t-a_n}{r_n})$ and $J_n(t) = J(\frac{t-a_n}{r_n})$. We define

$$v_n(t,x_1) = \exp(-q_n - z_n(t-a_n)) \cos \sqrt{z_n} x_1,$$
$$w_n(t,x_2) = \exp(-q_n - z_n(t-a_n) + J_n(t)p_n) \cos \sqrt{z_n} x_2,$$

and

$$u(t,x_1,x_2) = \begin{cases} v_1(t,x_1) & \text{for } t \le a_1, \\ A_n(t)v_n(t,x_1) + B_n(t)w_n(t,x_2) + C_n(t)v_{n+1}(t,x_1) & \text{for } a_n \le t \le a_{n+1}, \\ 0 & \text{for } t \ge 0. \end{cases}$$

If for all α, β $\gamma > 0$

$$\lim_n \exp(-q_n + 2p_n) z_{n+1}^{\alpha} p_n^{\beta} r_n^{-\gamma} = 0 \tag{4.4}$$

then u is a $C_b^\infty(\mathbb{R}^3)$ function. We define

$$l(t) = \begin{cases} 1 & \text{for } t \le a_1 \text{ or } t \ge 0, \\ 1 + J_n'(t)p_n z_n^{-1} & \text{for } a_n \le t \le a_{n+1}. \end{cases}$$

The condition

$$\sup_n \{p_n r_n^{-1} z_n^{-1}\} \le \frac{1}{2\|J'\|_{L^\infty}} \tag{4.5}$$

guarantees that the operator $\L = \partial_t - \partial_{x_1}^2 - l(t)\partial_{x_2}^2$ is parabolic. Moreover l is a C^μ function under the condition

$$\sup_n \{\frac{p_n r_n^{-1} z_n^{-1}}{\mu(r_n)}\} < +\infty. \tag{4.6}$$

Finally we define

$$b_1 = -\frac{\L u}{u^2 + (\partial_{x_1} u)^2 + (\partial_{x_2} u)^2} \partial_{x_1} u,$$
$$b_2 = -\frac{\L u}{u^2 + (\partial_{x_1} u)^2 + (\partial_{x_2} u)^2} \partial_{x_2} u,$$
$$c = -\frac{\L u}{u^2 + (\partial_{x_1} u)^2 + (\partial_{x_2} u)^2} u$$

and as in [15] the coefficients b_1, b_2, c wil be in C_b^∞ if for all α, β, $\gamma > 0$

$$\lim_n \exp(-p_n) z_{n+1}^\alpha p_n^\beta r_n^{-\gamma} = 0. \tag{4.7}$$

We choose

$$a_n = -\sum_{j=n}^{+\infty} \frac{1}{(j+k_0)^2 \mu(\frac{1}{j+k_0})}, \qquad z_n = (n+k_0)^3 \tag{4.8}$$

with k_0 sufficiently large. Changing t in $-t$ the proof of Theorem 3 will be complete as soon as under the choice (4.8) the conditions (4.1),..., (4.7) hold. Let's verify this. Since the function $\sigma \mapsto 1/(\sigma^2\mu(1/\sigma))$ is decreasing on $[1, +\infty[$ (see Remark 1) we have that the hypothesis (2.2) is equivalent to the convergence of the series $\sum_n ((n+k_0)^2 \mu(\frac{1}{n+k_0}))^{-1}$ and (4.1) follows. Condition (4.2) is obvious. We have, for $n \geq 2$,

$$q_n = \sum_{j=2}^{n} (j+k_0)^3 \frac{1}{(j+k_0-1)^2 \mu(\frac{1}{j+k_0-1})} \geq \sum_{j=2}^{n} \frac{j+k_0}{\mu(\frac{1}{j+k_0-1})}.$$

Remarking that $\mu(\frac{1}{j+k_0-1}) \leq 1$ we obtain that

$$q_n \geq \frac{1}{2}((n+k_0+1)(n+k_0) - (k_0+3)(k_0+2)) \tag{4.9}$$

for all $n \geq 2$. On the other hand

$$p_n = (3(n+k_0)^2 + 3(n+k_0) + 1) \frac{1}{(n+k_0)^2 \mu(\frac{1}{n+k_0})};$$

using also the fact that there exists $c > 0$ such that $\mu(s) \geq cs$ for all $s \in [0,1]$ we deduce that

$$\frac{3}{\mu(\frac{1}{n+k_0})} \leq p_n \leq \frac{3}{c}(n+k_0+2)$$

for all $n \geq 1$. Finally remarking that it is not restrictive to suppose that $\mu(s) \leq s^{1/2}$ for all $s \in [0,1]$ (if it is not so it is sufficient to replace $\mu(s)$ with $\min\{\mu(s),\ s^{\frac{1}{2}}\}$), we have

$$3(n+k_0)^{\frac{1}{2}} \leq p_n \leq \frac{3}{c}(n+k_0+2) \tag{4.10}$$

and

$$(n+k_0)^{-\frac{3}{2}} \le r_n \le \frac{1}{c}(n+k_0)^{-1} \tag{4.11}$$

for all $n \ge 1$. Easily the first part of (4.10) implies (4.3) if k_0 is sufficiently large and (4.9), (4.10) and (4.11) give (4.4) and (4.7). We observe that

$$\begin{aligned} p_n r_n^{-1} z_n^{-1} &= (z_{n+1} - z_n) z_n^{-1} \\ &= 3(n+k_0)^{-1} + 3(n+k_0)^{-2} + (n+k_0)^{-3} \le 7(n+k_0)^{-1} \end{aligned} \tag{4.12}$$

for all $n \ge 1$ and again taking k_0 is sufficiently large (4.5) follows. To prove (4.6) we start remarking that since the function $s \mapsto \frac{\mu(s)}{s}$ is decreasing on $]0,1]$ and $\lim_{s\to 0^+} \frac{\mu(s)}{s} = +\infty$ (see Remark 1) we have that there exists k_0 such that

$$r_n(n+k_0) = \frac{1}{(n+k_0)\mu(\frac{1}{n+k_0})} = \frac{\frac{1}{(n+k_0)}}{\mu(\frac{1}{n+k_0})} \le 1$$

for all $n \ge 1$, so that $r_n \le \frac{1}{n+k_0}$ and then

$$\frac{\mu(r_n)}{r_n} \ge \frac{\mu(\frac{1}{n+k_0})}{\frac{1}{(n+k_0)}} \tag{4.13}$$

for all $n \ge 1$. From (4.12) and (4.13) we have that

$$\frac{p_n r_n^{-1} z_n^{-1}}{\mu(r_n)} \le \frac{7}{n+k_0} \frac{1}{\mu(r_n)} \le \frac{7}{r_n(n+k_0)} \frac{r_n}{\mu(r_n)} \le 7 \, \frac{\mu(\frac{1}{n+k_0})}{\frac{1}{n+k_0}} \frac{1}{\frac{\mu(r_n)}{r_n}} \le 7.$$

The proof is complete.

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